Design of a Robust Estimator for Target Tracking

Eung T. Kim* and Dominick Andrisani II†
Purdue University, West Lafayette, Indiana 47907

I. Introduction

IN this Note, a new method for designing a constant gain estimator with reduced sensitivity to unknown variations in parameters of the plant is presented. A mechanism for trading off small increases in estimator error variance against low sensitivity to unknown parameter variations has been developed using the modified p norm of the error variances calculated at the discrete points in the range of the uncertain parameters. This robust estimator design method is applied to the target-tracking problem. In this example, an α-β tracking filter is designed to be robust to a stability derivative of the aircraft being tracked.

Target tracking or any other practical application of the Kalman filter is hindered by the fact that theoretical assumptions underlying the Kalman filter are not always valid. For instance, the Kalman filter requires exact knowledge of the plant model. In the example of tracking an unfriendly aircraft, the mathematical model of the target is usually unknown. Furthermore, the parameters of that mathematical model such as stability and control derivatives, might also be time varying. Designing a successful state estimator for a target tracker under these conditions is indeed a difficult task.

Haddad and Bernstein[1] developed a design method for robust state estimators that provide acceptable performance over the range of parametric uncertainty by choosing estimator gains to minimize the estimation error bound so that the actual estimation error is guaranteed to lie below the prescribed upper bound. Their design parameters are related to the structure of the uncertainty and it is not easy to choose the correct values of these design parameters.

An adaptive estimator can be used to reduce the sensitivity to parameter variations by occasionally updating the parameters of an estimator based on the latest identified parameters through on-line or off-line parameter identification techniques. However, the adaptation mechanism often places an excessive computational load on a real-time tracking algorithm.

H_∞ synthesis has emerged recently as a new design tool for the robust control system. H_∞ synthesis is a design method that minimizes the peak magnitude of the frequency response. The application of H_∞ synthesis to estimator design is described in Refs. 2 and 3, where the estimators were designed to be robust against the uncertainties in the input and initial conditions. This Note presents an alternative approach to robust estimator design using a different formulation for model uncertainty.

II. Problem Statement

The continuous, linear, time-invariant plant dynamics for a target aircraft can be described by the linear differential equations

$$\dot{x} = Ax + Bw$$

$$z = Hx \tag{1}$$

where A, B, and H are functions of system parameters a,z is an output vector that we want to estimate, and w is a zero-mean Gaussian white process noise vector with intensity Q. The system input is modeled here using process noise to approximate the unknown inputs applied to the aircraft by a pilot and atmospheric turbulences. The measurement output equation is given by

$$y = Cx + v \tag{2}$$

where v is a zero-mean Gaussian white measurement noise vector with intensity R.

We assume the estimator to be of the following form:

$$\dot{\boldsymbol{x}} = A_0\boldsymbol{x} + K(y - C_0\boldsymbol{x})$$

$$z = H_0\boldsymbol{x} \tag{3}$$

where $\boldsymbol{x}$ is an estimated state vector; A_0, C_0, and H_0 are estimator system matrices, and K is a gain vector of the estimator.

It is very common in the design of estimators (as in the target-tracking case) that the exact plant matrices A, B, H, and C are unknown to the designer of the estimator. Instead, the designer of the estimator must use a different, perhaps simpler, state-space model. We assume here that estimation system matrices A_0, H_0, and C_0 are chosen by the designer by whatever method he prefers. For example, the classical target-tracking models, such as α-β and α-β-γ trackers,[4] have A_0, B_0, and C_0 that are very simple in form and different from the A, B, and C of the vehicle being tracked. The design parameters for this problem are the estimator gains K. In selecting K, we are concerned about the sensitivity of the estimator to the parameter variations in the plant matrices A, B, and C.

III. Existing Solutions: Minisum and Minimax Methods

As a way of dealing with sensitivity to parameter variations in the controller design, the "expected cost" method was developed by Ly and Cannon[5] who used the expected value of the cost over the complete range of parameter variations as a cost function. Since the calculation of the expected value of the cost requires considerable computation for a continuous probability density function, the idea of using a finite sum of costs, each one evaluated at a fairly small number of points in the range of parameter variations, was considered.[6] This method of minimizing the sum of cost is called in this Note the "minisum" method.

The cost function in the minisum controller design may also be used for the estimator design and can be written as follows:

$$J(K) = \sum_{i=1}^{k} P_i J_i(K) \tag{4}$$

where P_i is the probability of $a = a_i$ and

$$J_i(K) = \text{tr} \lim_{t\to\infty} \left[E(e^TWe)\right]_{a=a_i}$$

where e is the estimation error $e \triangleq z - z$ and W a weighting matrix. With the minisum design procedure, the estimator gains K are chosen to minimize $J(K)$ in Eq. (4).

The "minimax" method is another design method frequently used when designing for robustness.[7] With the minimax design procedure, K would be chosen to minimize the largest $P_iJ_i(K)$. This approach requires the determination of the parameter giving the worst performance. This usually requires considerable computation.[6]

The minisum and minimax methods have inherent disadvantages. The result of the minisum method may be highly sensitive to parameter variations yielding very large values of performance indices at the worst condition even though the performance indices at the other conditions are very low,

---

Received June 18, 1990; presented as Paper 90-3401 at the AIAA Guidance, Navigation, and Control Conference, Portland, OR, Aug. 20-22, 1990; revision received April 1, 1991; accepted for publication July 7, 1992.

*Currently at the Flight Control Department, Korean Aerospace Research Institute, Teadok Science Town, Taejon, South Korea.

†Associate Professor, School of Aeronautics and Astronautics.

therefore allowing the sum of the performance indices to be the minimum. The $J(k)$ from the minimax method might be very large even though the performance index at the worst condition is the minimum we can achieve at that condition. This would correspond to a robust design having overall poor performance.

IV. New Solution: Mini-p-Norm Method

A new cost criterion that can avoid the foregoing problem is obtained by a simple modification of the cost function defined by Eq. (4). The idea is to give more weight to the large contributors in the performance index and less weight to the smaller ones. This can be done by choosing $(P_iJ_i)^{p-1}$ as a weight function, since $(P_iJ_i)^{p-1}$ is large when P_iJ_i is large and small when P_iJ_i is small for all $p>1$.

The new cost function can be written as follows:

$$J_A(K) = \sum_{i=1}^{k} [P_iJ_i]^p \tag{5}$$

It is the same as the p norm of the performance index vector J_c raised to the power p

$$J_A(K) = \left[\|J_c\|_p\right]^p$$

where $J_c \triangleq [P_1J_1 \quad P_2J_2 \quad \cdots \quad P_kJ_k]$

$$\|J_c\|_p \triangleq \left[\sum_{i=1}^{k} [P_iJ_i]^p\right]^{(1/p)}$$

Note that when $p=1$, this method is exactly the same as the minisum method, and for large p, the method approaches the minimax method. Since minimizing the aforementioned cost function is the same as minimizing the p norm of the performance index vector J_c, this new method will be called here a "mini-p-norm" method.

The algorithm to solve for K that minimizes $J_A(K)$ is discussed next. Equations (1-3) can be combined to yield

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ KA & A_0 - KC_0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$$

or

$$\dot{x}_a = A_a x_a + B_a u_a$$

where the subscript a stands for augmented matrices. From the definition of the error vector $e = Hx - H_0\hat{x}$, the cost function defined in Eq. (5) can be written as

$$J_A(K) = \sum_{i=1}^{k} \left[P_i \operatorname{tr}(X_iN_i)\right]^p$$

where the subscript i denotes the evaluation at $a = a_i$, and for each i we have

$$x \triangleq \lim_{t\to\infty} E\begin{bmatrix} xx^T & x\hat{x}^T \\ \hat{x}x^T & \hat{x}\hat{x}^T \end{bmatrix}, \quad N \triangleq \begin{bmatrix} H^TWH & -H^TWH_0 \\ -H_0^TWH & H_0^TWH_0 \end{bmatrix}$$

The steady-state value of X is given by the solution of the Lyapunov equation

$$A_aX + XA_a^T + B_aU_aB_a^T = 0 \tag{6}$$

where

$$U_a = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

The existence of a unique solution to Eq. (6) is guaranteed if the plant and estimator are both stable.[8]

The first-order necessary condition for the minimization of $J_A(K)$ is $[\mathrm{d}J_A(K)]/\mathrm{d}K = 0$. The gradient of $J_A(K)$ can be calculated by introducing a Lagrange multiplier matrix G, which satisfies the following Lyapunov equation:

$$A_a^TG + GA_a + \frac{\partial\left[P \operatorname{tr}(XN)\right]^p}{\partial X} = 0 \tag{7}$$

For each i, we can express G and X in 2×2 block matrices as follows:

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix}, \qquad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$$

If we assume constant R_i (i.e., $R_i = R$), K can be solved explicitly

$$K = -\left(\sum_{i=1}^{k} G_{22_i}\right)^{-1}\left\{\sum_{i=1}^{k}\left[G_{12_i}^T(X_{11_i}C_i^T - X_{12_i}C_0^T)\right.\right.$$
$$\left.\left. + G_{22_i}(X_{12_i}^TC_i^T - X_{12_i}C_0^T)\right]\right\}R^{-1} \tag{8}$$

where X_{11_i}, X_{12_i}, X_{22_i}, G_{12_i}, and G_{22_i} are calculated by Eqs. (6) and (7).

Since Eq. (8) and Eqs. (6) and (7) are coupled, the estimator gains K must be solved by iteration. An iterative relaxation approach[9] can be used here. First fix K and solve Eqs. (6) and (7) and then update K using Eq. (8). Repeat until convergence is achieved. Any Kalman filter gain can be used as an initial value for the iteration since it guarantees a stable estimator. For each iteration, the new value of K must stabilize the estimator. A minimum of $J_A(K)$ at each iteration will occur before a stability boundary is reached since $J_A(K)$ becomes extremely large as K approaches the stability boundary. Therefore, the estimator will remain stable by using a sufficiently small increment of ΔK for a new K.[10]

It can be shown that the mini-p-norm becomes the same as the standard Kalman filter if $A = A_0$, $B = B_0$, $C = C_0$, $H = H_0 = I$, $k = 1$, and $p = 1$. In this case, $M_{12} = -M_{22}$, and from Eqs. (11) and (12), it is clear that $G_{12} = -G_{22}$, which leads to $K = SC^TR^{-1}$ from Eq. (7), where $S \triangleq X_{11} - X_{12} - X_{12}^T - X_{22}$. From {Eq. (8) + Eq. (10)–Eq. (9)–[Eq. (9)]^T}, we obtain the following Riccati equation:

$$AS + SA - PC^TR^{-1}CP + BQB^T = 0$$

which is the same formula as for the standard Kalman filter.

V. Application to Aircraft Tracking

In this section we apply the insensitive estimator design methods previously described to the problem of tracking the vertical motion of an aircraft. The state of the system was defined as follows:

$$x = [u\ w\ q\ \theta\ h]^T$$

The system matrices A and B in Eqs. (1) and (2) were obtained from the longitudinal dynamic equation of the aircraft.[9] We want to estimate the altitude of the aircraft h, where h was assumed to be measured. Therefore, $C = H = [0\ 0\ 0\ 0\ 1]$. As an estimator model, the α-β filter was used that has the following structure[4]:

$$x = [h\ \dot{h}]^T$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_0 = [1\ \ 0]$$

The sensitivity to the change of stability derivative C_{m_α} of the aircraft was investigated. The trajectory of T-38 aircraft flying at a speed of 829 ft/s at an altitude of approximately 15,000 ft was estimated. The system input to the aircraft was assumed to be zero-mean Gaussian white noise, whose intensity was determined such that the variance of the acceleration in the vertical direction of each aircraft was $(2G)^2$ for all flight conditions (where G is the acceleration of gravity). The $2G$ was

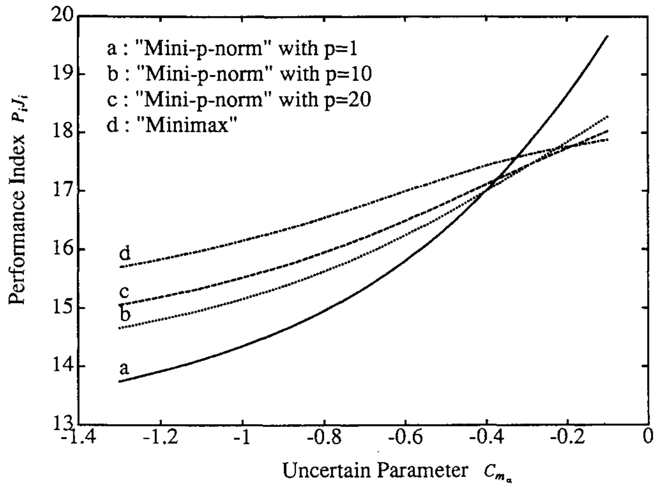

Fig. 1 **α-β estimator results for C_{m_α} variation of T-38.**

chosen to represent a moderately extensive pattern of jinking maneuvers. The noise intensity in the measurement of h was assumed to be 2.4 ft^2.

The C_{m_α} of the T-38 was changed from -1.3 to -0.1, which is the typical range of C_{m_α} of military aircraft. C_{m_α} is a term in the A matrix as specified in Ref. 11. Seven values of C_{m_α} equally spaced in this range were chosen as the quantization points ($k=7$), and the probability P_i was assumed to be the same for all cases of C_{m_α}.

The error variances of the estimator were calculated analytically by solving the Lyapunov equations. In Fig. 1, a–c are the results of the mini-p-norm method with $p=1$, 10, and 20, respectively, and d is the minimax estimator result, obtained from the Kalman filter designed at the worst flight condition (i.e., at $C_{m_\alpha}=-0.1$).

The sum of error variances shown as case a in Fig. 1 is the minimum we can achieve since the mini-p-norm method with $p=1$ is the same as the minisum method. As p increases, the maximum error variance decreases and the sensitivity to parameter variations is reduced, but at the cost of a larger sum of error variances. As p becomes too large, the maximum error variance decreases only very insignificant amounts, whereas the sum of error variances increases by substantial amounts. The mini-p-norm method with $p=10$ can be proposed as the most favorable design method in this tracking problem if low sensitivity and small error variance are both required. This illustrates the use of p as a design parameter in designing robust estimators.

VI. Conclusions

The mini-p-norm method for the design of estimators that are robust to parameter variations was presented. This method assumed that the estimator system matrices are fixed by the designer and finds the estimator gains minimizing the new cost criterion, which is directly related to the p norm of the performance index vector. This method has a new design parameter with which a tradeoff can be achieved between the expected value of error variance and sensitivity to parameter variations. If the value of this design parameter is equivalent to 1, our method becomes the minisum estimator, and as it becomes large, our method approaches the minimax method. The freedom to optimize the estimator using this design parameter gives the designer the option of avoiding the two extreme cases represented by the minisum and minimax methods and improving, it is hoped, the performance of his robust estimator. This method also can be used to design estimators that are robust to various aircraft types of flight conditions.

The same design algorithm developed here also can be used to find the estimator gains that would make the estimator insensitive to the variation of the maneuver intensity. In addition, the same design concept employed in this Note could be used to design a class of robust controllers that would have the same properties stated here, i.e., the capability of trading-off between the expected value of the cost function and the sensitivity to parameter variations using the new design parameter.

References

[1]Haddad, W. M., and Bernstein, D. S., "Robust, Reduced-Order, Nonstrictly Proper State Estimation via the Optimal Projection Equations with Guaranteed Cost Bounds," *IEEE Transactions on Automatic Control*, Vol. AC-33, June 1988, pp. 591–595.

[2]Haddad, W. M., and Bernstein, D. S., "Steady-State Kalman Filtering with an H_∞ Error Bound," *Systems & Control Letters*, Vol. 12, 1989, pp. 9–16.

[3]Khargonekar, P. P., and Nagpal, K. N., "Filtering and Smoothing in an H_∞ Setting," *Proceedings of 28th IEEE Conference on Decision and Control* (Tampa, FL), 1989, pp. 415–420.

[4]Fitzgerald, R. J., "Simple Tracking Filters: Closed Form Solutions," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-17, No. 6, 1981, pp. 781–785.

[5]Ly, U., and Cannon, R. H., Jr., "A Direct Method for Designing Robust Optimal Control Systems," *Proceedings of the AIAA Guidance and Control Conference* (Palo Alto, CA), AIAA, New York, Aug. 1978, pp. 440–448.

[6]Vinkler, A., and Wood, L. J., "A Comparison of Several Techniques for Designing Controllers of Uncertain Dynamic Systems," *Proceedings of the IEEE Conference on Decision and Control* (San Diego), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1978, pp. 31–38.

[7]Salmon, D. M., "Minimax Controller Design," *IEEE Transactions on Automatic Control*, Vol. AC-13, No. 4, 1968, pp. 369–376.

[8]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[9]Wagie, D. A., and Skelton, R. E., "Model Reduction and Controller Synthesis in the Presence of Parameter Uncertainty," *Automatica*, Vol. 22, No. 3, 1986, pp. 295–308.

[10]Choi, S. S., and Sirisena, H. R., "Computation of Optimal Output Feedback Gains for Linear Multivariable Systems," *IEEE Transactions on Automatic Control*, Vol. AC-19, June 1974, pp. 257–258.

[11]Smetana, F. O., *Computer Assisted Analysis of Aircraft Performance Stability and Control*, McGraw-Hill, New York, 1984.

Stabilizability of Linear Quadratic State Feedback for Uncertain Systems

Rong-Jyue Wang* and Wen-June Wang†
National Central University, Chung Li, 320, Taiwan, Republic of China

Introduction

In recent years, the problem of designing a stabilizing feedback control for a linear system containing time-varying uncertainties has received considerable attention. Based on Lyapunov's direct method, Cheres et al.[1] introduced a nonlinear controller to stabilize uncertain systems under the assumption that the system satisfies the so-called "matching conditions." In Thorp and Barmish,[2] linear controllers were derived to deal with the same problem. Recently, Tsay et al.[3] applied the conventional linear quadratic optimal state feedback method to find the robust regulator for linear uncertain systems with matching conditions. Moreover, Schmitendorf[4] used the Riccati equation approach to the design of a stabilizing controller for a class of uncertain linear systems without a matching condition.

In this Note, we consider the same problem but the uncertainties, which may exist on the system matrix and/or input matrix, are decomposed into the matching and mismatching

Received Feb. 25, 1992; revision received July 2, 1992; accepted for publication July 13, 1992.

*Graduate Student, Department of Electrical Engineering.
†Professor, Department of Electrical Engineering.